WHICH COMPACTA ARE NONCOMMUTATIVE ARS?

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Abstract. We give a short answer to the question in the title: *dendrits*. Precisely we show that the C^*-algebra $C(X)$ of all complex-valued continuous functions on a compactum X is projective in the category $\mathcal{C}^1$ of all (not necessarily commutative) unital C^*-algebras if and only if X is an absolute retract of dimension $\dim X \leq 1$ or, equivalently, that X is a dendrit.

1. Introduction

We recall that a compact space X is an *absolute retract* (AR) if for every injective continuous map $j : A \to Y$ and every continuous map $f\colon A \to X$ there exists a continuous extension, i.e., a map $\tilde{f}\colon Y \to X$ such that $\tilde{f} \circ j = f$.

$$\begin{tikzcd}
& Y \arrow[dl, dashed, "\tilde{f}"'] \\
X & A \arrow[l, "f"] \arrow[u, "j"']
\end{tikzcd}$$

In the dual language of the C^*-algebras of continuous complex-valued functions this means projectiveness of $C(X)$ in the category of commutative unital C^*-algebras. Namely, for any epimorphism of commutative C^*-algebras $p\colon B \to A$ and any *-homomorphism $f\colon C(X) \to A$, there is a lift $\tilde{f}\colon C(X) \to B$, $p \circ \tilde{f} = f$.

$$\begin{tikzcd}
& B \arrow[d, "p"] \\
C(X) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & A
\end{tikzcd}$$

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A compact space X is a *noncommutative AR* if $C(X)$ is a projective object in the category of all unitary C^*-algebras. Clearly, a noncommutative AR is an absolute retract in ordinary sense.

Generally, let $\mathcal{M}$ be a subcategory of the category of all C^*-algebras which is closed under quotients. We use $\mathcal{C}$ to denote the category of all C^*-algebras and $*$-homomorphisms and $\mathcal{C}^1$ to denote the subcategory of unital C^*-algebras and unital $*$-homomorphisms. Let also $\mathcal{AM}$ denote the full subcategory of $\mathcal{M}$ consisting of abelian C^*-algebras. Then a C^*-algebra $P \in \mathcal{M}$ is said to be projective in $\mathcal{M}$ if for any $B \in \mathcal{M}$, ideal $J \subseteq B$ and morphism $f\colon P \to A/J$, there exists a morphism $\tilde{f}\colon P \to B$ such that $f = \tilde{f} \circ \pi$, where $\pi\colon B \to B/J$ is a quotient morphism. Here is the corresponding diagram

$$\begin{tikzcd} & B \arrow[d, "\pi"] \\ P \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B/J \end{tikzcd}$$

Example 1.1. The following observations are well known::

(a) $\mathbb{C}$ is projective in $\mathcal{C}^1$ but not in $\mathcal{C}$;
(b) $C([0,1])$ is projective in $\mathcal{C}^1$;
(c) $C(X)$ is projective in $\mathcal{AC}^1$ if and only if X is a compact absolute retract;
(d) $C\left([0,1]]^2\right)$ is not projective in $\mathcal{C}^1$.
(e) $C_0((0,1])$ is projective in $\mathcal{C}$.

It is important to outline a proof of (d). Let u be the unilateral shift on the separable Hilbert space $\ell_2(\mathbb{N})$ and let $C^*(u)$ be the corresponding Toeplitz alebra, i.e. the C^*-subalgebra of $\mathbb{B}(\ell_2(\mathbb{N}))$ generated by u. It is known [4] that there is a short exact sequence

$$0 \longrightarrow \mathbb{K}(\ell_2(\mathbb{N})) \hookrightarrow C^*(u) \xrightarrow{\pi} C(S^1) \longrightarrow 0$$

The real and imaginary parts of $\pi(u)$ (commuting self-adjoint contraction in $C(S^1)$) determine a $*$-homomorphism $f\colon C([0,1]^2) \to C(S^1)$ which cannot be lifted to $C^*(u)$.

We note that first the notion of noncommutative ANR was introduced by Blackadar [1] which became known under the name of semiprojective (commutative) C^*-algebras [6], [7]. In [7] it is shown that every finite graph is a noncommutative ANR. Using his technique it is easy to show that every finite tree is a noncommutative AR.

2. Projectivity and liftable relations

Lemma 2.1. *Suppose that a metrizable compactum Y can be represented as the union $Y = X_1 \cup X_2$ of its connected closed subspaces. If $|X_1 \cap X_2| = 1$ and $C(X_k)$ is projective in $\mathcal{C}^1$ for each $k = 1, 2$, then $C(Y)$ is projective in $\mathcal{C}^1$.*

Proof. Let $Y_k = X_k \setminus (X_1 \cap X_2)$, $k = 1, 2$. Since X_k obviously is the one-point compactification of Y_k it follows (see, for instance, [7, Theorem 10.1.9]) that $C_0(Y_k)$ is projective in $\mathcal{C}$, $k = 1, 2$. By [7, Theprem 10.1.11], $C_0(Y_1 \cup Y_2) = C_0(Y_1) \oplus C_0(Y_2)$ is also projective in $\mathcal{C}$. Finally since Y is the one-point compactification of the sum $Y_1 \cup Y_2$ we conclude, again referring to [7, Theorem 10.1.9], that $C(Y)$ is projective in $\mathcal{C}^1$. □

Corollary 2.2. *Let X be a finite tree. Then $C(X)$ is projective in $\mathcal{C}^1$.*

Proof. Observe that $C([0, 1])$ is projective in $\mathcal{C}^1$ and repeatedly apply Lemma 2.1. □

We recall some definitions from [7]. Given a relation

$$\mathcal{R} \subset C^*\langle x_1, \dots, x_n \mid \|x_i\| \leq 1\rangle$$

its *representation* in a C^*-algebra A is an n-tuple of constructions $a_1, \dots a_n \in A$ such that $\Phi(p) = 0$ for all $p \in \mathcal{R}$ where

$$\Phi : C^*\langle x_1, \dots, x_n \mid \|x_i\| \leq 1\rangle \to A$$

with $\Phi(x_i) = a_i$. If only $\|\Phi(p)\| < \delta$ for all p, then it is called a *δ-representation* of $\mathcal{R}$ in A.

Let $(E, \leq)$ be finite partially ordered set with the property that each element has at most one predecessor. We denote by $\mathcal{R}(E)$ the following relation set:

$0 \leq e \leq 1$ *for* $e \in E$;

$(e - 1)e' = 0$ *if* $e \leq e'$, *and*

$ee' = 0$ *if* e *and* e' *are incomparable;* $e, e' \in E$.

This set of relations occurs on generators of the algebra $C(T)$ for a finite tree T. Under a tree we mean a connected graph without loops. By $V(T)$ and by $E(T)$ we denote the set of vertices and the set of edges respectively. Fixing a root in T gives the order on $E = E(T)$ by the rule: $e \leq e'$ if the shortest path to the root from e' uses e. It also defines the orientation on edges $e = [v_e^-, v_e^+]$ with v_e^- to be the closest to the root. Denote by h_e the distance to v_e^- function defined on e and extended to T by means of the natural collapse of $T \setminus e$ to the end points of e.

Proposition 2.3. *The family $\{h_e \mid e \in E(T)\}$ together with the constants $\mathbb{C}$ generate the algebra $C(T)$.*

Proof. Every function $f \in C(T)$ can be uniquely presented as the sum $f = f(o) + \sum_e f_e$ with $f_e = \phi_e r_e$ and $\phi_e \in C_0((v_e^-, v_e^+]) \cong C_0((0,1])$ where $o \in T$ denotes the root and $r_e : T \to e$ is the retraction collapsing the complement to the edge e to its end points. We show this by induction on the hight of T, the maximal length of branches. Certainly it is true for trees of hight 0, i.e., one point ($= o$). Assume that it holds true for trees of hight $< k$ and let T be of hight k. Then T can be presented as a tree T' of hight $k-1$ with a family of edges E' attached to vertices of T' with the distance $k-1$ from the root. By induction assumption $f|_{T'} = f(o) + \sum_{e\in E(T')} \phi_e r'_e$ where $r' : T' \to e$ is the retraction. Clearly, $f - (f(o) + \sum_{e\in E(T')} \phi_e r_e)$ is the sum of functions ϕ_e with supports in $e \in E'$. This implies existence of the presentation. Since each ϕ_e, $e \in E'$, is uniquely defined, we obtain the uniqueness.

Each function ϕ_e can be "expressed" in terms of h_e, since the function $h(t) = t$ generates $C_0((0,1])$. □

Note that $\{h_e \mid e \in E\}$ satisfies the relations $\mathcal{R}(E)$. We will refer to $\{h_e \mid e \in E(T)\}$ as to the *standard basis* of the algebra $C(T)$ for a rooted tree T.

A set of relations $\mathcal{R}$ on a set G is called *liftable* if, for any epimorphism of C^*-algebras $\pi : A \to B$ and a representation $< b_g >_{g\in G}$ in B there is a lifting to a representation $< a_g >_{g\in G}$ in A also satisfying $\mathcal{R}$ and such that $\pi(a_g) = b_g$. Then a projectivity of the universal C^*-algebra $C^*(G \mid \mathcal{R})$ is equivalent to the liftability of $\mathcal{R}$ (see [7] for more details). In view of this we can restate the Corollary 2.2 as follows.

Proposition 2.4. *For every finite tree T the relation set $\mathcal{R}(E(T))$, is liftable.*

Proof. We apply Lemma 3.2.2 of [7] to get that $C(T)$ is the universal algebra in $\mathcal{C}^1$ for the relation set $\mathcal{R}(E(T))$. □

We recall [7] that a finite relation is called *stable* if for every $\epsilon > 0$ there is $\delta > 0$ such that for every epimorphism $\pi : A \to B$ and every δ-representation $(x_1, \dots, x_n)$ of $\mathcal{R}$ in A such that $(\pi(x_1), \dots, \pi(x_n))$ is a representation for $\mathcal{R}$ in B, there is a representation $(y_1, \dots, y_n)$ for $\mathcal{R}$ in A such that $\|y_i - x_i\| < \epsilon$ and $\pi(y_i) = \pi(x_i)$.

Since the stability of relations means exactly the semiprojectivity of the universal algebra and projectivity implies semiprojectivity we can conclude (see Theorem 14.1.4 [7]) that the following holds true:

Proposition 2.5. *The relations $\mathcal{R}(E(T))$ are stable for any finite tree T.*

3. Topological preliminaries

The following proposition might be well-known.

Proposition 3.1. *Let X be a Peano continuum of dimension > 1. Then X contains a topological copy of the circle S^1.*

Proof. We present a proof based on Borsuk's theorem which states that every Peano continuum X admits a geodesic metric d. It means that for every pair of points $x, x' \in X$ there is an isometric imbedding of the interval $\xi : [0, a] \to X$ with $a = d(x, x')$, $\xi(0) = x$, and $\xi(a) = x'$. The image $\xi([0, a])$ is called a *geodesic segment* between x and x' and is denoted by $[x, x']$.

Assume that X does not contain a circle and $\dim X > 1$. The first condition implies that for every two pints $x, x' \in X$ there is a unique geodesic joining them. Moreover, every piece-wise geodesic path between x and x' contains the geodesic segment $[x, x']$.

Since $indX > 1$, there is $x_0 \in X$ and $r > 0$ such that $\dim \partial S_r(x_0) > 0$ where $S_r(x_0) = \{x \in X \mid d(x, x_0) = r\}$ is the sphere of radius r centered at x_0. Then $S_r(x_0)$ contains a continuum C. Let $y_0, y_1 \in C$ and let $z \in [y_0, x_0] \cap [y_1, x_0]$ be the point with the maximum $d(x_0, z)$. We denote by $I = [y_0, z] \cup [z, y_1]$. Thus, $I = [y_0, y_1]$. Let $\epsilon = r - d(x_0, z)$. We consider a finite cover of C by $\epsilon/4$-balls. Since C is a continuum, the nerve of this cover is connected. Therefore, there is a finite sequence $z_0, z_1, \dots, z_k \in C$ such that $z_0 = y_0$, $z_k = y_1$, and $d(z_i, z_{i=1}) < \epsilon$. Clearly, $z \notin [z_i, z_{i+1}]$ for every i. This contradicts to the fact that a piece-wise geodesic path $[z_0, z_1] \cup [z_1, z_2] \cup \cdots \cup [z_{k-1}, z_k]$ contains I. □

Proposition 3.2. *Let $X \in AR$ be a compact Hausdorff space of dimension > 1. Then X contains a topological copy of the circle S^1.*

Proof. Scepin's theorem about the adequate correspondence between compact ARs and soft maps [9],[3] allows to reduce the problem to the case when X is metrizable AR compactum. Indeed, by Schepin's theorem there is a soft map $p : X \to X_\alpha$ onto a metrizable AR compactum X_α of the same dimension. We take a topological circle $S^1 \subset X_\alpha$ and lift it to X. The possibility of lifting is a part of the definition of soft maps. □

REMARK. The Proposition 3.2 holds true for compact Hausdorff AE(1) compacta. In this case one should apply the adequate correspondence theorem from [5] (see also [3]). We recall that $AE(n)$ stands for absolute extensors for the class of n-dimensional spaces, i.e., such spaces Y that every extension problem has a solution in case $\dim X \leq n$.

4. The main theorem

For a compact space X and a point $x \in X$ we denote by $C_x(X) = C_0(X \setminus \{x\})$ the C^*-algebra of a locally compact space $X \setminus \{x\}$.

Let $T' = T \cup I$ be a tree obtained from a tree T by attaching an edge $I = [v, w]$ to a vertex. We identify $C(T)$ and $C(I)$ with the subalgebras of $C(T')$ by means of corresponding collapses.

Proposition 4.1. *Let $\pi : B \to A$ be a surjection of unital C^*-algebras and let $\phi : C(T') \to A$ be a C^*-morphism. Then for any lift $\xi : C(T) \to B$ of $\phi|_{C(T)}$*

and any $\epsilon > 0$ there is a lift $\xi' : C(T') \to B$ of ϕ such that $\|\xi(h_e) - \xi'(h_e)\| < \epsilon$ where $\{h_e\}_{e\in E(T)}$ is the standard basis of $C(T)$.

Proof. Let $\xi : C(T) \to B$ and $\epsilon > 0$ be given. Since the relations $\mathcal{R}(E(T))$ are stable there is $\delta > 0$ that serves ϵ. Consider the closed δ-ball $B_\delta(v)$ in T with respect to the graph metric on T. Let $q : T \to T$ be a map that collapses the ball $B_{\delta/2}(v)$ fixes $T \setminus B_\delta(v)$ and linearly extends to $B_\delta(v) \setminus B_{\delta/2}(v)$. Let $w_e = q^*(h_e)$, $e \in E(T)$. Then $\|w_e - h_e\| < \delta$ in $C(T')$ and hence $\|\xi(w_e) - \xi(h_e)\| < \delta$ in B.

Let $u \in C_v(T)$, $0 \le u \le 1$, be such that $gu = g$ for every $g \in q^*(C_v(T))$. Let h denote the generator of $C_0((v,w]) \subset C(T')$. Let $\bar{h} \in B$ be an arbitrary lift of h with $\|\bar{h}\| \le 1$. We define $\tilde{h} = \bar{h} - \xi(u)\bar{h}$. Note that $\|\tilde{h}\| \le \|\bar{h}\|\|1-u\| \le 1$. For every $g \in q^*C_v(T)$ we have

$$\xi(g)\tilde{h} = \xi(g)(\bar{h} - \xi(u)\bar{h}) = \xi(g)\bar{h} - \xi(gu)\bar{h} = \xi(g)\bar{h} - \xi(g)\bar{h} = 0.$$

We show that $\{\xi(h_e)\}_{e\in E(T)} \cup \{\tilde{h}\}$ is a δ-representation in B of the relations $\mathcal{R}(E(T'))$. First, we note the inequality part of relations holds true. Also the relations that do not involve I holds true. If $e \le I$ then $h_e - 1 \in C_v(T)$ and hence $(\xi(w_e) - 1)\tilde{h} = 0$. Hence $\|(\xi(h_e) - 1)\tilde{h}\| =$

$$\|(\xi(h_e) - 1)\tilde{h} - (\xi(w_e) - 1)\tilde{h}\| = \|(\xi(h_e) - \xi(w_e))\tilde{h}\| \le \|\xi(h_e) - \xi(w_e)\| < \delta.$$

If e and I are not comparable, then $\xi(w_e)\tilde{v} = 0$ and similarly, $\|(\xi(h_e)\tilde{h}\| < \delta$.

In view of stability (Proposition 2.5) there is a presentation $(y_e)_{e\in E(T)} \cup \{y_I\}$ in B of the relations $\mathcal{R}(E(T'))$ with $\pi(y_e) = \phi(h_e)$, $\pi(e_I) = h$, $\|y_e - \xi(h_e)\| < \epsilon$, $e \in E(T)$, and $\|y_I - \tilde{h}\| < \epsilon$. We define $\xi' : C(T_k) \to B$ by setting $\xi'(h_e) = y_e$, $e \in E(T')$. □

Proposition 4.2. *Let a tree T' be obtained from a tree T by adding an extra vertex in the middle of an edge $e \in E(T)$. Thus $e = e_- \cup e_+$. Let $\xi, \psi : C(T) \to A$ be such that $\|\xi(h) - \psi(h)\| < \epsilon$ for all elements of the new standard basis $\{h_b\}_{b\in E(T')}$. Then the inequality $\|\xi(h) - \psi(h)\| < \epsilon$ for all elements of the old standard basis $\{h_a\}_{a\in E(T)}$.*

Proof. Since $h_e = \frac{1}{2}(h_{e_-} + h_{e_+})$ in $C(T)$, the result follows. □

Theorem 4.3. *The following conditions are equivalent for a compact space X:*

(1) *$C(X)$ is projective in $\mathcal{C}^1$;*
(2) *X is an absolute retract and* $\dim X \le 1$.

Proof. (1) $\Longrightarrow$ (2). If $C(X)$ is projective in $\mathcal{C}^1$ then it is projective in the smaller category $\mathcal{AC}^1$. By the Gelfand duality, the latter is equivalent to X being an absolute retract. In order to prove that $\dim X \le 1$, assume the contrary, i.e. suppose that $\dim X > 1$. Then by Proposition 3.2 X contains a topological copy of the circle S^1. Let $i \colon S^1 \hookrightarrow X$ denote the corresponding embedding.

By the Gelfand duality the $*$-homomorphism $f\colon C([0,1]^2) \to C(S^1)$ (see the proof of Example 1.1(d)) is of the form $f = C(j)$ for embedding map $j\colon S^1 \to [0,1]^2$. Since $[0,1]^2$ an absolute retract there exists a map $g\colon X \to [0,1]^2$ such that $g \circ i = j$. This implies that $C(i) \circ C(g) = C(j) = f$. In other words the following diagram of unbroken arrows

$$\begin{tikzcd}
 & & C^*(u) \arrow[d, "\pi"] \\
C([0,1]^2) \arrow[rr, "C(j)=f"] \arrow[dr, "C(g)"'] & & C(S^1) \\
 & C(X) \arrow[ur, "C(i)"'] \arrow[uur, dashed, "\varphi"] &
\end{tikzcd}$$

commutes. Since $C(X)$ is projective in $\mathcal{C}^1$, the $*$-homomorphism $C(i)$ can be lifted to a $*$-homomorphism (the dotted arrow in the above diagram) $\varphi\colon C(X) \to C^*(u)$. Then

$$\pi \circ (\varphi \circ C(g)) = (\pi \circ \varphi) \circ C(g) = C(i) \circ C(g) = C(j) = f$$

which shows that the $*$-homomorphism f also has a lifting contradicting our choice. Consequently $\dim X \leq 1$.

(2) $\Longrightarrow$ (1). Let X be a dendrit. Thus, X is the inverse limit of finite trees T_k with bonding maps $r_k : T_{k+1} \to T_k$ be the retraction which takes I_k to the attaching point $x_k = T_k \cap I_k$, $T_{k+1} = T_k \cup I_k$, $T_0 = I_0 \cong [0,1]$, $I_k = [x_k, y_k] \cong [0,1]$. Let $\rho_k : T_{k+1} \to I_k$ be the retraction which takes T_k to the point x_k. Let $C = C(X)$, $C_k = C(X_k)$ and $A_k = C(I_k)$. The maps r_k and ρ_k induce imbeddings r_k^* of C_k and ρ_k^* of A_k into C_{k+1}. Let $h_k \in C_0((x_k, y_k]) \cong C_0((0,1])$ be the generator. The image of h_k under this imbedding (as well as under composition imbeddings $r_{k+l}^* \circ \cdots \circ r_{k+1}^* \circ r_k^*$) will be denoted by the same symbol h_k.

Thus, $C = \lim_{\to}\{C_k, r_k^*\}$ is the direct limit. Since all bonding maps are imbeddings, we regard C_k as a subalgebra of C for all k. Let $\pi : B \to C$ be an epimorphism. We define sections $\psi_k : C_k \to B$ for all k such that $\psi_{k+1}|_{C_k} = \psi_k$. Then the direct limt of ψ_k will define a required section.

By induction on k we construct sections $\xi_k : C_k \to W$. Since $C(T_0)$ is projective, there is a section ξ_0. Assume that ξ_k is constructed. We construct ξ_{k+1} using Propostion 4.1 with $\epsilon = 1/2^k$.

Let $\{h_e^k\}_{e \in E(T_k)}$ be the standard basis for C_k defined by the rooted tree structure on T_k with the root $0 \in [0,1] = I_0 = T_0$. Fix $e \in E(T_k)$. By induction on i in

view of Proposition 4.2 and Proposition 4.1 we obtain $\|\xi_{k+i}(h_e^k) - \xi_{k+i-1}(h_e^k)\| \leq 1/2^{k+i}$ for every $i \in \mathbb{N}$. Therefore for every k and $e^k \in E(T_k)$ there is a limit

$$\lim_{i\to\infty} \xi_{k+i}(h_e^k) = \bar{h}_e^k.$$

We define $\psi_k(h_e^k) = \bar{h}_e^k$. This defines a presentation of the relation set $\mathcal{R}(E(T_k))$ in B and hence a homomorphism of C^*-algebras $\psi_k : C_k \to B$. Note that ψ_k is a lift. Also note that $\psi_{k+1}(h_e^k) = \bar{h}_e^k = \psi_k(h_e^k)$ if $e \in E(T_{k+1})$. If $e \notin E(T_{k+1})$, it means that $e = e_- \cup e_+$ in T_{k+1} and $h_e^k = \frac{1}{2}(h_{e_-}^{k+1} + h_{e_+}^{k+1})$ (see Proposition 4.2). Then

$$\psi_{k+1}(h_e^k) = \frac{1}{2}\psi_{k+1}(h_{e_-}^{k+1}) + \frac{1}{2}\psi_{k+1}(h_{e_+}^{k+1}) = \frac{1}{2}\lim_{i\to\infty} \xi_{k+i}(h_{e_-}^{k+1})$$

$$+\frac{1}{2}\lim_{i\to\infty} \xi_{k+i}(h_{e_+}^{k+1}) = \lim_{i\to\infty} \xi_{k+i}(\frac{1}{2}(h_{e_-}^{k+1} + h_{e_+}^{k+1})) = \lim_{i\to\infty} \xi_{k+i}(h_e^k) = \psi_k(h_e^k).$$

Thus, $\psi_{k+1}(g) = \psi_k(g)$ for all $g \in C_k$. □

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